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On integrable Doebner–Goldin equations

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Abstract. We integrate sub-families of a family of nonlinear Schrödinger equations proposed by Doebner and Goldin in the $(1+1)$-dimensional case which have exceptional Lie symmetries. Since the method of integration involves non-local transformations of dependent and independent variables, general solutions obtained include implicitly determined functions. By properly specifying one of the arbitrary functions contained in these solutions, we obtain broad classes of explicit square integrable solutions. The physical significance and some analytical properties of the solutions obtained are briefly discussed.

1. Introduction

The semi-direct product of the group of diffeomorphisms and the Abelian group of smooth functions on $\mathbb{R}^n$ may be regarded as a *generalized symmetry group* on $\mathbb{R}^n$. From the representation theory of this group, Doebner and Goldin derived a family of nonlinear Schrödinger equations on $\mathbb{R}^n$ [1–4], which have been called the Doebner–Goldin (DG) equations [5] (for recent progress in the study of these equations see the contributions in [6]):

$$\mathrm{i}\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + V(\boldsymbol{x})\right)\psi + \mathrm{i}\tfrac{1}{2}\hbar D R_2[\psi]\psi + \hbar D' \sum_{j=1}^{5} c_j R_j[\psi]\psi \qquad (1)$$

where $R_j[\psi]$, $j = 1, \ldots, 5$ are real-valued functionals of the density $\rho := \psi\bar{\psi}$ and the current $\boldsymbol{J} = \mathrm{Im}(\bar{\psi}\boldsymbol{\nabla}\psi)$,

$$\begin{aligned} &R_1[\psi] := \frac{\boldsymbol{\nabla}\cdot\boldsymbol{J}}{\rho} = \mathrm{Im}\,\frac{\Delta\psi}{\psi} \qquad R_2[\psi] := \frac{\Delta\rho}{\rho} = \frac{\Delta(|\psi|^2)}{|\psi|^2} \\ &R_3[\psi] := \frac{\boldsymbol{J}^2}{\rho^2} = \left(\mathrm{Im}\,\frac{\boldsymbol{\nabla}\psi}{\psi}\right)^2 \qquad R_4[\psi] := \frac{\boldsymbol{J}\cdot\boldsymbol{\nabla}\rho}{\rho^2} = \mathrm{Im}\left(\frac{\boldsymbol{\nabla}\psi}{\psi}\right)^2 \\ &R_5[\psi] := \frac{(\boldsymbol{\nabla}\rho)^2}{\rho^2} = \left(\frac{\boldsymbol{\nabla}(|\psi|^2)}{|\psi|^2}\right)^2. \end{aligned} \qquad (2)$$

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Here the real number D (with the physical dimension of a diffusion constant) labels unitarily inequivalent representations of the generalized symmetry group involved in the derivation of the nonlinear equations (1). It has been interpreted as a quantum number describing dissipative quantum systems [3, 4]. The real number D' (also with the physical dimension of a diffusion constant) describes the magnitude of the real nonlinearity and the dimensionless constants $c_j \in \mathbb{R}$ are 'model' parameters.

For the purpose of this paper it is more convenient to use the parametrization that is obtained by rewriting equation (1) in terms of the real functionals $R_j[\psi]$ only, following the notation of [7–10]:

$$F(\nu,\mu) : \mathrm{i}\partial_t\psi = \mathrm{i}\sum_{j=1}^{2}\nu_j R_j[\psi]\psi + \sum_{j=1}^{5}\mu_j R_j[\psi]\psi + \mu_0 V\psi \qquad \nu_1 \neq 0. \tag{3}$$

Particular homogeneous equations of this type have also been considered in the context of quantum mechanics by other authors, e.g. [11–17]

One of the interesting features of the family of DG equations (1) is its invariance under a certain group of transformations [7, 8]

$$N_{(\Lambda,\gamma)}(\psi) = \psi^{\frac{1}{2}(1+\Lambda+\mathrm{i}\gamma)}\bar{\psi}^{\frac{1}{2}(1-\Lambda+\mathrm{i}\gamma)} = |\psi|\mathrm{e}^{\mathrm{i}(\gamma\ln|\psi|+\Lambda\arg\psi)} \tag{4}$$

i.e. if ψ is a solution of $F(\nu,\mu)$, then $\psi' = N_{(\Lambda,\gamma)}(\psi)$ is a solution of $F(\nu',\mu')$, where the change of parameters under $N_{(\Lambda,\gamma)}$ is

$$\begin{aligned}
&\nu_1' = \frac{\nu_1}{\Lambda} \qquad \nu_2' = -\frac{\gamma}{2\Lambda}\nu_1 + \nu_2 \qquad \mu_1' = -\frac{\gamma}{\Lambda}\nu_1 + \mu_1 \\
&\mu_2' = \frac{\gamma^2}{2\Lambda}\nu_1 - \gamma\nu_2 - \frac{\gamma}{2}\mu_1 + \Lambda\mu_2 \qquad \mu_3' = \frac{\mu_3}{\Lambda} \\
&\mu_4' = -\frac{\gamma}{\Lambda}\mu_3 + \mu_4 \qquad \mu_5' = \frac{\gamma^2}{4\Lambda}\mu_3 - \frac{\gamma}{2}\mu_4 + \Lambda\mu_5 \qquad \mu_0' = \Lambda\mu_0.
\end{aligned} \tag{5}$$

Thus, without loss of generality we can restrict our calculations to the particular choice of parameters (a particular *gauge*, see below)

$$\nu_1 = -1 \qquad \nu_2 = 0. \tag{6}$$

Since the transformations (4) leave the position probability invariant, i.e. $\rho'(\boldsymbol{x}t) = \rho(\boldsymbol{x}t)$, they have been called *nonlinear gauge transformations* [7, 8]. This notion is physically motivated by the fact that in (non-relativistic) quantum mechanics we basically measure positions at different times. Furthermore, the transformations have been used to construct a consistent notion of observables in a nonlinear quantum theory [18].

It turned out that besides such important properties of the DG equation as homogeneity, separability and Euclidean invariance, which were 'input' by construction, equations (1) possess a number of other attractive properties. Among them one should emphasize the possibility of constructing explicit square integrable solutions, which is important for a physical interpretation. In particular, some stationary and non-stationary (Gaussian and travelling wave) solutions have been obtained [4, 5, 9, 19–21].

The well known connection between exact solutions of partial differential equations (PDEs) and their symmetry properties [22–24] as well as the necessity of classifying equations (1) in a unified way, motivated a systematic study of their Lie symmetry in [10]. As a result, one has to distinguish nine sub-families (characterized by conditions on the parameters μ in the chosen gauge) with different maximal Lie symmetry algebras $\mathrm{sym}_{\cdot}(n)$. The relationship between these sub-families and their symmetries is indicated in figure 1 (using the notation of [10]).

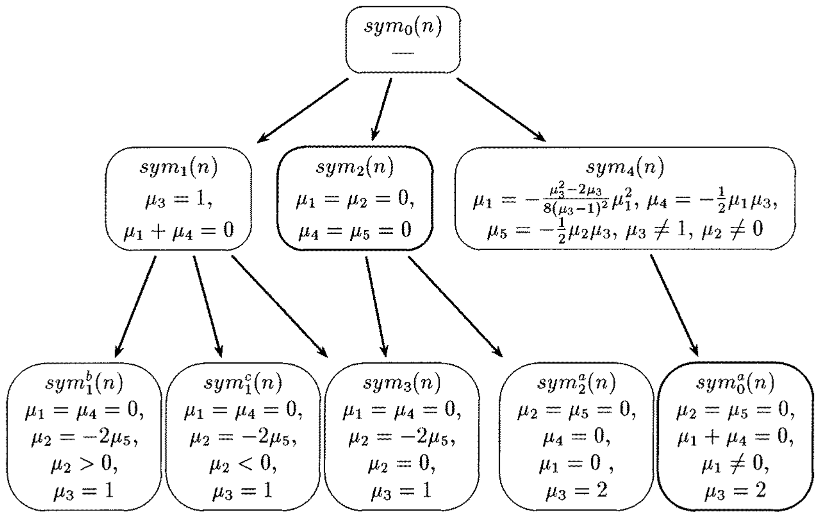

Figure 1. Lie symmetries of the DG equation. Sub-families are characterized by their parameters and arrows indicate the subfamily structure. The equations dealt with in this paper are in bold frames.

Five of these symmetry algebras are finite dimensional, $\mathrm{sym}_0(n)$–$\mathrm{sym}_4(n)$. Among them ($\mathrm{sym}_1(n)$) we find the direct sum of the (centrally extended) Schrödinger algebra and the real numbers (due to real homogeneity of the equations). These equations thus fit into the classes of Schrödinger invariant nonlinear evolution equations determined in [25–28].

The four remaining symmetry algebras are infinite dimensional. $\mathrm{sym}_1^b(n)$ and $\mathrm{sym}_1^c(n)$ contain in addition to the elements of $\mathrm{sym}_1(n)$ infinite-dimensional algebras g_b and g_c, that depend on a pair of (real) solutions of a linear forward and backward heat equations and a (complex) solution of a linear Schrödinger equation, respectively. In fact these symmetries correspond to linearization of these sub-families, the first to a pair of forward and backward heat equations, the latter to a Schrödinger equation [29, 30]. In contrast, the symmetry algebras $\mathrm{sym}_2^a(n)$ and $\mathrm{sym}_0^a(n)$ contain an infinite-dimensional algebra g_a that depends only on one real-valued function. As a consequence, there is no local transformation (i.e. a transformation that does not involve integrals or derivatives of the dependent variables) linearizing the corresponding DG equations. Nevertheless, these equations as well as the one admitting the finite-dimensional symmetry algebra $\mathrm{sym}_3(n)$ are shown in the present paper to be integrable by a *non-local* transformation of dependent and independent variables in the case of one spatial variable ($n = 1$). Thus, *all* DG equations with exceptional symmetries (bottom row of figure 1) sym_1^b, sym_1^c, sym_3, sym_2^a, sym_0^a are *integrable*, i.e. they can be reduced to an equation which is either linear or integrable by quadratures.

The principal object of study in the present paper is the DG equation in $(1 + 1)$ dimensions with parameters

$$\nu_1 = -1 \qquad \nu_2 = 0 \qquad \mu_1 = \mu_2 = \mu_4 = \mu_5 = 0 \tag{7}$$

$$\nu_1 = -1 \qquad \nu_2 = 0 \qquad \mu_2 = \mu_5 = 0 \qquad \mu_3 = 2 \qquad \mu_4 = -\mu_1 \neq 0 \tag{8}$$

i.e. the following coupled two-dimensional PDEs:

$$\mathrm{i}\psi_t = \left\{-\mathrm{i}\,\mathrm{Im}\,\frac{\psi_{xx}}{\psi} + \mu_3\left(\mathrm{Im}\,\frac{\psi_x}{\psi}\right)^2 + \mu_0 V(x)\right\}\psi \tag{9}$$

$$\mathrm{i}\psi_t = \left\{(\mu_1 - \mathrm{i})\,\mathrm{Im}\,\frac{\psi_{xx}}{\psi} + 2\left(\mathrm{Im}\,\frac{\psi_x}{\psi}\right)^2 - \mu_1\,\mathrm{Im}\left(\frac{\psi_x}{\psi}\right)^2 + \mu_0 V(x)\right\}\psi. \tag{10}$$

One of these DG equations is contained in the so-called Ehrenfest sub-family [4, 30] fulfilling the second Ehrenfest relation: equation (9) with $\mu_3 = 1$, i.e. the DG equation with maximal Lie symmetry $\mathrm{sym}_3(n)$. This equation is, furthermore, the only Schrödinger and Galilei invariant equation among (9) and (10). Nevertheless, in the free case ($V \equiv 0$) all of these DG equations admit travelling (solitary) wave solutions with arbitrary shape [9]. These solutions are rediscovered as a particular case of the general solutions in this paper.

Using a polar decomposition

$$\psi(x,t) = \exp\left(r(x,t) + \mathrm{i}s(x,t)\right) \tag{11}$$

we rewrite the above equations in the following way:

$$F_1 : \begin{cases} r_t + s_{xx} + 2r_x s_x = 0 \\ s_t + \mu_3 s_x^2 = -\mu_0 V \end{cases} \tag{12}$$

$$F_2 : \begin{cases} r_t + s_{xx} + 2r_x s_x = 0 \\ s_t + \mu_1 s_{xx} + 2s_x^2 = -\mu_0 V. \end{cases} \tag{13}$$

Let us note that the second equation of system (12) is nothing but the one-dimensional Hamilton–Jacobi equation, which is integrated either by contact transformation [31] or by the method of characteristics [31, 32]. The second equation of system (13) is the potential Burgers equation which is transformed to the standard Burgers equation by means of differentiation with respect to x and a subsequent change $s_x \to u$. The Burgers equation received much attention, since it describes a wide range of physical phenomena and is known to be linearizable by the Cole–Hopf substitution [33, 34]. After the pioneering work of Burgers [35] the mathematical theory of shock waves was developed (see [36, 37] and references therein).

What is remarkable about systems (12) and (13) is that it is possible to construct the general solutions of the first equations for arbitrary solutions of the second ones.

The paper is organized as follows. In section 2 we integrate the free equations, i.e. equations (12) and (13) with a vanishing potential ($V \equiv 0$). In order to integrate F_1 we have to distinguish between the cases $\mu_3 \neq 1$ and $\mu_3 = 1$, the latter corresponding to the sub-family with the larger Lie symmetry algebra $\mathrm{sym}_3(n) \supset \mathrm{sym}_2(n)$.

Section 3 contains some remarks on the integration of the equations with potential and two particular examples where the integration is carried out. The methods of integration of F_1 and F_2 in sections 2 and 3 yield their general solutions containing an implicitly determined function. Consequently, these solutions are, generally speaking, implicit. Therefore, in section 4 we give some explicit solutions for the free equation as well as for linear and quadratic potentials by specifying one of the arbitrary functions of the general solutions obtained in the preceding sections.

2. Integration of free DG equations

Putting in (12), (13) $V = 0$ we obtain the following PDEs:

$$\tilde{F}_1 : \begin{cases} r_t + s_{xx} + 2r_x s_x = 0 \\ s_t + \mu_3 s_x^2 = 0 \end{cases} \tag{14}$$

$$\tilde{F}_2 : \begin{cases} r_t + s_{xx} + 2r_x s_x = 0 \\ s_t + \mu_1 s_{xx} + 2s_x^2 = 0. \end{cases} \tag{15}$$

Henceforth we suppose that in (15) $\mu_1 \neq 0$, since otherwise system (15) is a particular case of (14) with $\mu_3 = 2$.

2.1. Integration of the family $\widetilde{F}_1$

First, we turn to the integration of the system of nonlinear PDEs (14). As this system admits only a finite-dimensional Lie symmetry group, there is no local transformation which linearizes it. So the only possibility of transforming the system in question into an integrable form is to utilize a non-local transformation of dependent and independent variables.

Differentiating the second equation of (14) with respect to x and denoting

$$R(x,t) = r(x,t) \qquad S(x,t) = s_x(x,t) \tag{16}$$

we get

$$R_t + 2SR_x + S_x = 0 \qquad S_t + 2\mu_3 S S_x = 0. \tag{17}$$

We can apply the method of characteristics to integrate the second equation of system (17) (see, e.g., [31, 32]). As a result, we obtain the function $S(t,x)$ in implicit form

$$F(\underbrace{x - 2\mu_3 t S}_{\omega_1}, \underbrace{S}_{\omega_2}) = 0 \tag{18}$$

with some sufficiently smooth function F.

Since (18) is an implicit equation, solutions $S(x,t)$ will, in general, be defined only locally in x and t. Indeed, the only globally $(x, t \in \mathbb{R})$ defined smooth (C^1) solutions are constant, and solutions defined for positive times have to be monotonically non-decreasing in x. More general global solutions are discontinuous and lead to the non-uniqueness of the initial value problem; consequently, one has to use additional arguments in order to select the 'physically relevant' global solutions. The usual way is to employ the 'entropy method' providing uniqueness of the solution of the initial value problem [37]. Here we will proceed to construct local (smooth) solutions and we will not consider how to extend these solutions outside of their domain of definition. In section (4.1) there will be examples of local solutions and solutions defined for positive t and $x \in \mathbb{R}$.

Provided $\partial F/\partial\omega_1 \not\equiv 0$, we can solve (18) locally for ω_1, thus getting

$$x = 2\mu_3 t S + f(S) \tag{19}$$

where $f \in C^2(\mathbb{R}, \mathbb{R})$ is an arbitrary function.

If on the contrary $\partial F/\partial\omega_1 \equiv 0$ holds, then by (18) $S =$ constant. Evidently $S =$ constant solves the second equation of (17) and, furthermore, is not contained in the class (19). That is why we have to distinguish the cases $S \neq$ constant and $S =$ constant and consider these separately.

Case 1. $S \neq \textit{constant}$. With the above condition the general solution of the second equation of the system (17) is given by formula (19). We look for the general solution of the first equation in the form

$$R = R(S,t) \tag{20}$$

which is always possible since due to the second equation of (17) $S_x \not\equiv 0$.

Inserting (20) into the first equation of (17) yields a first-order linear PDE with non-constant coefficients,

$$R_t + \frac{2(1-\mu_3)S}{2\mu_3 t + f'(S)} R_S = -\frac{1}{2\mu_3 t + f'(S)}. \tag{21}$$

When integrating the above equation we have to distinguish two sub-cases $\mu_3 = 1$ and $\mu_3 \neq 1$. Let us recall that the DG equation with parameters (7) and $\mu_3 = 1$ satisfies the Ehrenfest relation and, what is more, admits an additional symmetry operator (see figure 1).

Sub-case 1.1 $\mu_3 = 1$. In this case equation (21) takes the form

$$R_t = -\frac{1}{2t + f'(S)}$$

and its general solution reads

$$R(S,t) = -\tfrac{1}{2}\ln(f'(S) + 2t) + g(S) \qquad g \in C^2(\mathbb{R}, \mathbb{R}). \tag{22}$$

Returning to the initial variables $(x, t, r(x,t), s(x,t))$ yields the general solution of the system of PDEs (14) with $s_x \neq$ constant, $\mu_3 = 1$

$$\begin{cases} r(x,t) = -\frac{1}{2}\ln(f'(S) + 2t) + g(S) \\ s(x,t) = \displaystyle\int_0^x S(\xi,t)\,\mathrm{d}\xi - \int_0^t S^2(0,\tau)\,\mathrm{d}\tau \end{cases} \tag{23}$$

where $f, g \in C^2(\mathbb{R}, \mathbb{R})$ are arbitrary functions and $S = S(x,t)$ is determined implicitly by (19) with $\mu_3 = 1$.

Sub-case 1.2. $\mu_3 \neq 1$. Using in (21) the transformation

$$R(S,t) = \tilde{R}(S,t) - \frac{1}{2(1-\mu_3)}\ln S$$

we get the homogeneous equation

$$\tilde{R}_t - \frac{2(1-\mu_3)S}{2\mu_3 t + f'(S)}\tilde{R}_S = 0.$$

The above PDE is integrated by means of the standard method of characteristics. Omitting the details of integration we give its general solution

$$R(S,t) = \frac{1}{2(\mu_3 - 1)}\ln S + g\left(2(\mu_3 - 1)tS^{\mu_3/(\mu_3-1)} + \int^S \zeta^{1/(\mu_3-1)} f'(\zeta)\,\mathrm{d}\zeta\right) \tag{24}$$

where $f, g \in C^2(\mathbb{R}, \mathbb{R})$ are arbitrary functions.

Returning to the initial variables x, t, r, s we obtain the general solution of the DG equation (14) for the case $s_x \neq$ constant, $\mu_3 \neq 1$

$$\begin{cases} r(x,t) = \dfrac{1}{2(\mu_3 - 1)}\ln S + g\left(2(\mu_3 - 1)tS^{\mu_3/(\mu_3-1)} + \displaystyle\int^S \zeta^{1/(\mu_3-1)} f'(\zeta)\,\mathrm{d}\zeta\right) \\ s(x,t) = \displaystyle\int_0^x S(\xi,t)\,\mathrm{d}\xi - \mu_3\int_0^t S^2(0,\tau)\,\mathrm{d}\tau \end{cases} \tag{25}$$

where $f, g \in C^2(\mathbb{R}, \mathbb{R})$ are arbitrary functions and the function $S = S(x,t)$ is determined implicitly by (19).

Case 2. $S = constant$. In this case $s(x,t) = C_1 x + \beta(t)$ with a constant C_1 and an arbitrary smooth function $\beta(t)$. Substituting this expression into (14) we arrive at

$$r_t + 2C_1 r_x = 0 \qquad \beta'(t) + \mu_3 C_1^2 = 0.$$

An integration of these equations gives rise to the following expressions for r and s:

$$\begin{cases} r(x,t) = f(x - 2C_1 t) \\ s(x,t) = C_1 x - \mu_3 C_1^2 t + C_2 \end{cases} \tag{26}$$

where C_1, C_2 are arbitrary real constants. Thus we have rediscovered the travelling (solitary) wave solutions with arbitrary shape [9] as a particular case of the general solution.

We have established that any smooth solution is contained (at least locally) in one of the classes given by (23), (25) and (26). Summarizing we arrive at the conclusion that the general solution of the free DG equation (9) splits into two inequivalent classes:

(1)

$$\psi(x,t) = f(x - 2C_1 t)\exp\{\mathrm{i}(C_1 x - \mu_3 C_1^2 t + C_2)\} \tag{27}$$

where C_1, C_2 are arbitrary real parameters, and

(2)

$$\psi(x,t) = \begin{cases} (f'(S) + 2t)^{-\frac{1}{2}} g(S) \\ \qquad \times \exp\left\{\mathrm{i}\left(\int_0^x S(\xi,t)\,\mathrm{d}\xi - \int_0^t S^2(0,\tau)\,\mathrm{d}\tau\right)\right\} & \mu_3 = 1 \\ S^{1/2(\mu_3-1)} g\left(2(\mu_3 - 1)t S^{\mu_3/(\mu_3-1)} + \int^S \zeta^{1/(\mu_3-1)} f'(\zeta)\,\mathrm{d}\zeta\right) \\ \qquad \times \exp\left\{\mathrm{i}\left(\int_0^x S(\xi,t)\,\mathrm{d}\xi - \mu_3\int_0^t S^2(0,\tau)\,\mathrm{d}\tau\right)\right\} & \mu_3 \neq 1 \end{cases} \tag{28}$$

where f, g are arbitrary, sufficiently smooth functions and $S = S(x,t)$ is determined implicitly by (19).

Although these formulae give the general solution of the corresponding DG equation for all μ_3, the case $\mu_3 = 0$ deserves special consideration, as the general solution of the system of PDEs (14) with $\mu_3 = 0$ can be obtained in explicit form. The second equation of (14) yields that s does not depend on time, so

$$s(x,t) = f(x) \qquad f \in C^2(\mathbb{R},\mathbb{R}).$$

Inserting this into the first equation we get a first-order PDE for r,

$$r_t + 2f'(x)r_x + f''(x) = 0. \tag{29}$$

The case $f(x) = \text{constant}$ leads to a travelling wave solution (27) with $\mu_3 = 0, C_1 = 0$; if $f'(x) \not\equiv 0$, then the general solution reads

$$r(x,t) = g\left(2t - \int^x \frac{\mathrm{d}\xi}{f'(\xi)}\right) - \tfrac{1}{2}\ln f'(x) \tag{30}$$

where $g \in C^2(\mathbb{R},\mathbb{R})$ is again an arbitrary function.

Consequently, the general solution of the DG equation (9) with $\mu_3 = 0$ is either given by a travelling wave solution (27) with $\mu_3 = 0, C_1 = 0$, or by

$$\psi(x,t) = (f'(x))^{-\frac{1}{2}} g\left(2t - \int^x \frac{\mathrm{d}\xi}{f'(\xi)}\right)\exp\{\mathrm{i}f(x)\}. \tag{31}$$

2.2. *Integration of the family* $\widetilde{F}_2$

Let us turn to the integration of the DG equation (15). First, as noted above, the second equation is the potential Burgers equation, which is is linearized by logarithmic substitution. Furthermore, we reduce the order of spatial derivatives in the first equation by a linear transformation of the dependent variables. Thus, the transformation

$$v(x,t) = -\mu_1 r(x,t) + s(x,t) \qquad u(x,t) = \exp\left(\frac{2}{\mu_1} s(x,t)\right) \tag{32}$$

reduces system (15) to the form

$$uv_t + \mu_1 u_x v_x = 0 \qquad u_t + \mu_1 u_{xx} = 0. \tag{33}$$

The second equation may be taken as the integrability condition of the vector field $(u, -\mu_1 u_x)$ on spacetime, $\partial_t u = \partial_x(-\mu_1 u_x)$, so that it is the gradient of a smooth function $\varphi(x,t)$,

$$\varphi_t = -\mu_1 u_x \qquad \varphi_x = u. \tag{34}$$

With this remark the first equation of the system (33) is rewritten to be

$$\varphi_x v_t - \varphi_t v_x = 0$$

and is easily integrated $v(x,t) = f(\varphi(x,t))$, where f is an arbitrary, sufficiently smooth function.

Solving (34) with respect to φ we get

$$\varphi(x,t) = \int_0^x u(\xi,t)\,\mathrm{d}\xi - \mu_1 \int_0^t u_x(0,\tau)\,\mathrm{d}\tau + C \tag{35}$$

where C is an arbitrary constant.

Returning to the initial variables $(r(x,t), s(x,t))$ we get the general solution of the DG equation (10)

$$\psi(x,t) = (u(x,t))^{\frac{1}{2}} f\left(\int_0^x u(\xi,t)\,\mathrm{d}\xi - \mu_1 \int_0^t u_x(0,\tau)\,\mathrm{d}\tau\right) \exp\left\{\frac{\mathrm{i}\mu_1}{2} \ln u(x,t)\right\} \tag{36}$$

where $u(x,t)$ is an arbitrary solution of the heat equation

$$u_t + \mu_1 u_{xx} = 0 \tag{37}$$

and f is an arbitrary smooth function.

Finally, we note that the travelling wave solutions of [9] are reobtained using the particular solution

$$u(x,t) = \exp\left\{-\frac{\alpha}{\mu_1}(x - \alpha t)\right\}$$

of the heat equation (37).

3. DG equation with non-vanishing potential

Surprisingly enough, DG equations (12), (13) are integrated in quadratures even in the case when $V(x) \neq 0$ (i.e. in the presence of a non-vanishing potential). Unfortunately, for the family F_1 the corresponding formulae are implicit and cumbersome. That is why we restrict ourselves to considering in detail system (12) with an additional constraint $\mu_3 = 1$ (i.e. the Ehrenfest sub-family of (12) is studied); this system was also considered in [29].

3.1. Integration of the family F_1

Choosing in (12) $\mu_3 = 1$, we obtain the following system of PDEs:

$$r_t + 2r_x s_x + s_{xx} = 0 \qquad s_t + s_x^2 + \mu_0 V(x) = 0. \tag{38}$$

Differentiating the second equation with respect to x and making the change of dependent variables (16) we arrive at a system of first-order PDEs:

$$R_t + 2SR_x + S_x = 0 \qquad S_t + 2SS_x + \mu_0 V'(x) = 0. \tag{39}$$

A solution of the above system is looked for in the form

$$t = u(x, S) \qquad R = v(x, S). \tag{40}$$

This transformation is defined in any domain where $S_t \not\equiv 0$. So again we have to distinguish two cases, $S_t \not\equiv 0$ and $S_t \equiv 0$.

Case 1. $S_t \not\equiv 0$. Inserting (40) into (39) we get

$$\mu_0 V'(x) u_S - 2S u_x + 1 = 0 \qquad \mu_0 V'(x) v_S - 2S v_x + \frac{u_x}{u_S} = 0.$$

A further change of variables

$$\tilde{u}(x, S) = u(x, S) \qquad \tilde{v}(x, S) = v(x, S) + \tfrac{1}{2} \ln u_S(x, S) \tag{41}$$

transforms the system to

$$\mu_0 V'(x) \tilde{u}_S - 2S\tilde{u}_x + 1 = 0 \qquad \mu_0 V'(x) \tilde{v}_S - 2S\tilde{v}_x = 0. \tag{42}$$

Thus, combining local and non-local transformations of the dependent and independent variables we linearized and decoupled the differential consequence of (38). Integrating these equations yields

$$\begin{aligned} \tilde{u}(x, S) &= \tfrac{1}{2} \int^x (S^2 + \mu_0(V(x) - V(\zeta)))^{-\frac{1}{2}} \,\mathrm{d}\zeta + f(S^2 + \mu_0 V(x)) \\ \tilde{v}(x, S) &= g(S^2 + \mu_0 V(x)) \end{aligned} \tag{43}$$

where $f, g \in C^2(\mathbb{R}, \mathbb{R})$ are arbitrary functions.

Returning to the variables $(x, t, R(x, t), S(x, t))$ we obtain the general solution of system (39) in an implicit form

$$t = \tfrac{1}{2} \int^x (S^2(x, t) + \mu_0(V(x) - V(\xi)))^{-\frac{1}{2}} \,\mathrm{d}\xi + f(S^2(x, t) + \mu_0 V(x)) \tag{44}$$

$$R(x, t) = \tfrac{1}{2} \ln S_t(x, t) + g(S^2(x, t) + \mu_0 V(x)). \tag{45}$$

Finally, rewriting these equations in the initial dependent variables $(r(x, t), s(x, t))$ we get the general solution of the initial DG equation

$$\begin{cases} r(x, t) = -\frac{1}{2} \ln S_t(x, t) + g(S^2(x, t) + \mu_0 V(x)) \\ s(x, t) = \displaystyle\int_0^x S(\xi, t) \,\mathrm{d}\xi - \int_0^t S^2(0, \tau) \,\mathrm{d}\tau - \mu_0 V(0) t + C \end{cases} \tag{46}$$

where $S(x, t)$ is a smooth function determined implicitly by (44) and f, g are arbitrary sufficiently smooth functions.

Case 2. $S_t \equiv 0$. With this condition the system of PDEs (39) is easily integrated to yield

$$R(x,t) = -\tfrac{1}{4}\ln(C_1 - \mu_0 V(x)) + \left(2t - \int^x \frac{\mathrm{d}\xi}{\sqrt{C_1 - \mu_0 V(\xi)}}\right)$$
$$S(x,t) = \sqrt{C_1 - \mu_0 V(x)}$$

where g is an arbitrary sufficiently smooth function and C_1 is an arbitrary real constant.

Rewriting the above expressions in the initial variables $(r(x,t), s(x,t))$ we get

$$\begin{cases} r(x,t) = -\tfrac{1}{4}\ln(C_1 - \mu_0 V(x)) + g\left(2t - \displaystyle\int^x \frac{\mathrm{d}\xi}{\sqrt{C_1 - \mu_0 V(\xi)}}\right) \\ s(x,t) = \displaystyle\int_0^x \sqrt{C_1 - \mu_0(\xi)}\,\mathrm{d}\xi - C_1 t + C_2 \end{cases} \tag{47}$$

where C_2 is an arbitrary constant.

Thus, we have established that the general solution of the DG equation (9) with $\mu_3 = 1$ splits into the following two classes:

(1)

$$\psi(x,t) = (S_t(x,t))^{\frac{1}{2}} g(S^2(x,t) + \mu_0 V(x)) \times \exp\left\{\mathrm{i}\left(\int_0^x S(\xi,t)\,\mathrm{d}\xi - \int_0^t S^2(0,\tau)\,\mathrm{d}\tau - \mu_0 V(0)t + C\right)\right\} \tag{48}$$

where $g \in C^2(\mathbb{R},\mathbb{R})$, $C \in \mathbb{R}$ and $S(x,t)$ is determined implicitly by (44) and

(2)

$$\psi(x,t) = (C_1 - \mu_0 V(x))^{-\frac{1}{4}} g\left(2t - \int_0^x \frac{\mathrm{d}\xi}{\sqrt{C_1 - \mu_0 V(\xi)}}\right) \times \exp\left\{\mathrm{i}\left(\int_0^x \sqrt{C_1 - \mu_0 V(\xi)}\,\mathrm{d}\xi - C_1 t + C_2\right)\right\} \tag{49}$$

where $g \in C^2(\mathbb{R},\mathbb{R})$ and $C_1, C_2 \in \mathbb{R}$.

3.2. Integration of the family F_2

Integrating system (13) with potentials is similar to integrating the free system ($V(x) = 0$) in section 2.2. Using the change of variables (32) for (13) we arrive at the following system of PDEs for new functions $u(x,t)$ and $v(x,t)$:

$$u_t + \mu_1 u_{xx} + \frac{2\mu_0}{\mu_1} V(x) u = 0 \qquad u v_t + \mu_1 u_x v_x + \mu_0 V(x) u = 0. \tag{50}$$

Now, given a potential $V(x)$ and an arbitrary solution $u(x,t)$ of the first equation of this system, one can construct a general solution $v(x,t)$ of the second equation which leads to a solution of the initial system:

$$\psi(x,t) = (u(x,t))^{\frac{1}{2}} \exp\left(-\mu_1^{-1} v(x,t) + \mathrm{i}\frac{\mu_1}{2}\ln u(x,t)\right). \tag{51}$$

4. Explicit solutions

As mentioned before some of the obtained solutions of the DG equation are *local* in the sense that they are not determined on the whole plane $\mathbb{R}^2$. However, for physical applications one needs *global* solutions and, what is more, they should be square integrable, i.e. the integral

$$p = \int_{-\infty}^{\infty} \bar{\psi}(x,t)\psi(x,t)\,\mathrm{d}x \tag{52}$$

is to be finite. If it is, the quantity $\rho(x,t) = (1/p)\bar{\psi}(x,t)\psi(x,t) \geqslant 0$ is treated as a probability density of a distribution of the wavefunction ψ in space at a given time.

4.1. Explicit solutions of the family F_1

Evidently, the travelling wave solutions of the free DG equation (9) given by (27) are defined on the whole plane and, consequently, are global. To ensure square integrability of these solutions one has to restrict the choice of the arbitrary function f to square integrable ones,

$$p = \int_{-\infty}^{\infty} f^2(\tau)\,\mathrm{d}\tau < \infty.$$

Thus, the travelling wave solutions are square integrable provided f is.

Solutions (28) are, generally speaking, local, since the function $S(x,t)$ contained in these solutions is determined implicitly by formula (19) and the existence of a solution is only guaranteed locally by the implicit function theorem. In order to obtain explicit expressions for global and strictly local solutions we consider solutions (28) with linear and quadratic functions f, respectively.

For linear functions f,

$$f(S) = -2\mu_3\alpha S \qquad \mu_3 \neq 0$$

the implicit equation (19) for S can be solved globally and we get $S(x,t) = x/2\mu_3(t-\alpha)$. Thus, we arrive at the following class of explicit solutions of the DG equation (9) for $\mu_3 \neq 0$ containing an arbitrary smooth function g:

$$(t-\alpha)^{-1/2\mu_3} g\left((t-\alpha)^{-1/\mu_3}x\right) \exp\left\{\frac{\mathrm{i}x^2}{4\mu_3(t-\alpha)}\right\}. \tag{53}$$

These solutions are square integrable, provided g is, and are well defined on the whole plane $\mathbb{R}^2$ with the possible exception of the line $t = \alpha$ for $\mu_3 > 0$, where they converge to a δ-function at the origin for suitable functions g. Note that in this case the wavepacket spreads out for $t \to \infty$ in the sense that the probability of finding the system in any finite interval converges to zero,

$$\lim_{t\to\infty} \int_a^b \bar{\psi}(x,t)\psi(x,t)\,\mathrm{d}x = 0 \qquad \forall a, b \in \mathbb{R}. \tag{54}$$

Thus the behaviour of the solutions for large times is similar to that of solutions of the linear Schrödinger equation. A notable difference is the behaviour near $t = \alpha$; unlike the linear case, there is a whole family of solutions (labelled by g) with the δ-function as an initial distribution at $t = \alpha$. For $\mu_3 < 0$ the behaviour of the wavepacket is reversed, i.e. it spreads out for $t \to \alpha$ and converges to a delta function in the origin for $t \to \infty$. In view of a quantum mechanical interpretation of the DG equation (9), this may be a reason to reject the case of positive μ_3.

In particular for $g(z) = \exp(-z^2)$ solutions (53) coincide with the Gaussian waves given explicitly for $\mu_3 = 1$ in [20].

A quadratic function f,

$$f(S) = S^2$$

gives rise to strictly local solutions of the DG equation. Indeed, inserting it into (19) we obtain a quadratic equation with respect to S

$$S^2 + 2\mu_3 t S - x = 0. \tag{55}$$

It has real solutions in the case $x + \mu_3^2 t^2 \geqslant 0$ only, i.e. $S(x,t)$ is not defined inside the parabola $x + \mu_3^2 t^2 = 0$. Solving (55) yields $S(x,t) = -\mu_3 t \pm \sqrt{x + \mu_3^2 t^2}$; according to the general solutions (28) we have to choose the positive sign since $2t + f'(S)$ ($\mu_3 = 1$) and S ($\mu_3 \neq 1$), respectively, have to be positive. Hence, according to (28), we arrive at the following class of strictly local explicit solutions of the family F_1 (9) containing an arbitrary smooth function g:

$$\psi(x,t) = \begin{cases} \left(\sqrt{x + \frac{t^2}{4}} - \frac{t}{2}\right)^{-1} g\left(2\ln\left(\sqrt{x + \frac{t^2}{4}} - \frac{t}{2}\right) - t\left(\sqrt{x + \frac{t^2}{4}} - \frac{t}{2}\right)^{-1}\right) & \\ \qquad \times \exp\left\{\frac{\mathrm{i}}{3}\left(2\left(x + \frac{t^2}{4}\right)\left(\sqrt{x + \frac{t^2}{4}} - \frac{t}{2}\right) + \frac{t}{2}x\right)\right\} & \mu_3 = \frac{1}{2} \\ \left(\sqrt{x + \mu_3^2 t^2} + (\mu_3 - 1)t\right)^{-1/2\mu_3} g\left(\left(\sqrt{x + \mu_3^2 t^2} - \mu_3 t\right)\right. & \\ \qquad \times \left.\left(\sqrt{x + \mu_3^2 t^2} + (\mu_3 - 1)t\right)^{(\mu_3 - 1)/\mu_3}\right) & \\ \qquad \times \exp\left\{\frac{1}{3}\mathrm{i}(2(x + \mu_3^2 t^2)\left(\sqrt{x + \mu_3^2 t^2} - \mu_3 t\right) + \mu_3 t x)\right\} & \mu_3 \neq \frac{1}{2}. \end{cases} \tag{56}$$

The domain of definition of these solutions is the set $\{(x,t) : x + \mu_3^2 t^2 > 0\}$. Furthermore, as the function $\sqrt{x + \mu_3^2 t^2}$ is not defined for x at $-\infty$ at any given time t, $|t| < \infty$, solution (56) is not square integrable.

In this context let us remark that in general solutions (28) are square integrable at a given time t provided

(i) the (possibly infinite) limits $a_\pm = \lim_{x\to\pm\infty} S(x,t)$ exist and

(ii) g is square integrable on the interval $[a_-, a_+]$.

This statement follows from a change of the integration variable $x \to S(x,t)$.

Before turning to DG equations with non-vanishing potentials we examine solution (31) of the particular case $\mu_3 = 0$ of the family F_1 that we have omitted before. If the first derivative of f has no zeros, then the solution given by (31) is certainly global. Again, a change of the integration variable shows that the solution is square integrable, provided that

(i) the (possibly infinite) limits $a_\pm = \lim_{\pm\infty} \int_0^x \mathrm{d}\tau / f'(\tau)$ exist and

(ii) g is square integrable on the interval $[a_-, a_+]$.

In the case of non-vanishing potentials we concentrate on the following specific potentials:

(1) the linear potential

$$V(x) = \frac{a}{\mu_0} x \qquad a \in \mathbb{R} \tag{57}$$

(2) the harmonic oscillator potential

$$V(x) = \frac{a^2}{\mu_0} x^2 \qquad a \in \mathbb{R} \tag{58}$$

(3) the anti-harmonic oscillator potential

$$V(x) = -\frac{a^2}{\mu_0} x^2 \qquad a \in \mathbb{R}. \tag{59}$$

First we consider the Ehrenfest case $\mu_3 = 1$, the integration of which has been studied in detail in section 3.1.

(1) For linear potentials (57) the implicit equation (44) reads

$$t = -\frac{1}{a} S(x,t) + f(S^2(x,t) + ax). \tag{60}$$

If we choose $f \equiv 0$, then $S(x,t) = -at$. Thus, we get a class of explicit solutions from (48):

$$\psi(x,t) = g(x + at^2) \exp\left\{-\mathrm{i}\left(atx + \frac{a^2}{3}t^3 - C\right)\right\}. \tag{61}$$

These solutions are defined on the whole plane $\mathbb{R}^2$ and square integrable, provided g is.

Another class of explicit solutions is obtained directly by means of formula (49):

$$\psi(x,t) = (C_1 - ax)^{-\frac{1}{4}} g\left(at + \sqrt{C_1 - ax}\right) \exp\left\{\mathrm{i}\left(-\frac{2}{3a}(C_1 - ax)^{\frac{3}{2}} - C_1 t + C_2\right)\right\} \tag{62}$$

where g is an arbitrary twice continuously differentiable function, C_1, C_2 are arbitrary parameters. These solutions are only defined on the half-plane $\{(x,t) : x < C_1/a\}$.

Analogously we construct explicit solutions for the (anti-)harmonic oscillator potentials. We give these without derivation. (g is an arbitrary sufficiently smooth function, C_1, C_2 are arbitrary parameters.)

(2) Harmonic oscillator potential (58)

$$\psi(x,t) = (\sin 2at)^{-\frac{1}{2}} g\left(\frac{x}{\sin 2at}\right) \exp\left\{\mathrm{i}\left(\frac{a}{2}x^2 \cot 2at + C_1\right)\right\} \tag{63}$$

$$\psi(x,t) = (C_1^2 - a^2x^2)^{-\frac{1}{4}} g\left(\sqrt{C_1^2 - a^2x^2}\sin 2at - ax\cos 2at\right)$$
$$\times \exp\left\{\mathrm{i}\left(\frac{x}{2}\sqrt{C_1^2 - a^2x^2} + \frac{C_1^2}{2a}\arcsin\frac{ax}{C_1} - C_1^2 t + C_2\right)\right\} \tag{64}$$

(3) Anti-harmonic oscillator potential (59)

$$\psi(x,t) = (\sinh 2at)^{-\frac{1}{2}} g\left(\frac{x}{\sinh 2at}\right) \exp\left\{\mathrm{i}\left(\frac{a}{2}x^2 \coth 2at + C_1\right)\right\} \tag{65}$$

$$\psi(x,t) = (C_1 + a^2x^2)^{-\frac{1}{4}} g\left(\left(ax + \sqrt{C_1 + a^2x^2}\right)\mathrm{e}^{-2at}\right)$$
$$\times \exp\left\{\mathrm{i}\left(\frac{x}{2}\sqrt{C_1 + a^2x^2} + \frac{C_1}{2a}\ln\left|ax + \sqrt{C_1 + a^2x^2}\right| - C_1 t + C_2\right)\right\}. \tag{66}$$

As mentioned in section 3.1, general solutions of the family F_1 (9) with $\mu_3 \neq 1$ are given by cumbersome implicit formulae. But with the particular choice of the potentials above it has explicit solutions containing one arbitrary function $g \in C^2(\mathbb{R}, \mathbb{R})$ (and a constant $C \in \mathbb{R}$):

Case I: $\mu_3 = 0$.

(1) Linear potential (57):

$$\psi(x,t) = g(x + at^2)\exp\{-\mathrm{i}(atx + C)\} \tag{67}$$

(2) harmonic oscillator potential (58):

$$\psi(x,t) = \mathrm{e}^{a^2t^2} g(x\mathrm{e}^{2a^2t^2})\exp\{-\mathrm{i}(a^2tx^2 + C)\} \tag{68}$$

(3) anti-harmonic oscillator potential (59):

$$\psi(x,t) = \mathrm{e}^{-a^2t^2} g\left(x\mathrm{e}^{-2a^2t^2}\right)\exp\{\mathrm{i}(a^2tx^2 + C)\}. \tag{69}$$

Case II: $\mu_3 = \lambda^2 > 0$.

(1) Linear potential (57):

$$\psi(x,t) = g(x + at^2)\exp\left\{-\mathrm{i}\left(atx + \frac{a^2\lambda^2}{3}t^3 + C\right)\right\} \tag{70}$$

(2) harmonic oscillator potential (58):

$$\psi(x,t) = (\sin 2a\lambda t)^{-1/2\lambda^2} g\left(\frac{x}{(\sin 2a\lambda t)^{1/\lambda^2}}\right)\exp\left\{\mathrm{i}\left(\frac{a}{2\lambda}x^2\cot 2a\lambda t + C\right)\right\} \tag{71}$$

(3) anti-harmonic oscillator potential (59):

$$\psi(x,t) = (\sinh 2a\lambda t)^{-1/2\lambda^2} g\left(\frac{x}{(\sinh 2a\lambda t)^{1/\lambda^2}}\right)\exp\left\{\mathrm{i}\left(\frac{a}{2\lambda}x^2\coth 2a\lambda t + C\right)\right\}. \tag{72}$$

Case III: $\mu_3 = -\lambda^2 < 0$.

(1) Linear potential (57):

$$\psi(x,t) = g(x + at^2)\exp\left\{-\mathrm{i}\left(atx - \frac{a^2\lambda^2}{3}t^3 + C\right)\right\} \tag{73}$$

(2) harmonic oscillator potential (58):

$$\psi(x,t) = (\sinh 2a\lambda t)^{1/2\lambda^2} g\left(\frac{x}{(\sinh 2a\lambda t)^{-1/\lambda^2}}\right)\exp\left\{-\mathrm{i}\left(\frac{a}{2\lambda}x^2\coth 2a\lambda t + C\right)\right\} \tag{74}$$

(3) anti-harmonic oscillator potential (59):

$$\psi(x,t) = (\sin 2a\lambda t)^{1/2\lambda^2} g\left(\frac{x}{(\sin 2a\lambda t)^{-1/\lambda^2}}\right)\exp\left\{-\mathrm{i}\left(\frac{a}{2\lambda}x^2\cot 2a\lambda t + C\right)\right\}. \tag{75}$$

When the linear potential solutions are similar to the travelling wave solutions of the free case; the shape of the wavepacket is conserved and 'moves' with constant acceleration $2a$ due to the constant force.

For the harmonic oscillator potential and $\mu_3 > 0$ the solutions are oscillating with the frequency $\omega = 2\sqrt{\mu_3}a$ and for $t \to n\pi/\omega$, $n \in \mathbb{N}$ they converge to δ-functions just like the fundamental solution of the linear Schrödinger equation with this potential (see e.g. [38, p 63]); but—as in the free case—there is a whole class of solutions for the DG equation (9) that converge to δ-functions periodically!

Similarly for the anti-harmonic oscillator potential and $\mu_3 > 0$; we again have a whole class of solutions, where a initial δ-function at $t = 0$ spreads exponentially in time.

The roles of the harmonic and the anti-harmonic oscillator are exchanged for negative μ_3.

In the boundary case $\mu_3 = 0$ the effect of the (anti-)harmonic oscillator is lost; the solutions of the harmonic oscillator converge to a δ-function for $t \to \infty$, whereas the solutions of the anti-harmonic oscillator do so for $t \to -\infty$.

If we choose $g(z) = \exp(-z^2)$ we re-obtain the Gaussian wave solutions discussed in [9, 20].

4.2. Explicit solutions of the family F_2

Clearly, if the function $u(x,t)$ is a global solution of the heat equation (37) then the formula (36) gives a global solution of the DG equation (10). And what is more, it is square integrable provided

(i) the (possibly infinite) limits $a_\pm = \lim_{\pm\infty}\left(\int_0^x u(\tau,t)\,\mathrm{d}\tau - \mu_1\int_0^t u_x(0,\tau)\,\mathrm{d}\tau\right)$ exist and

(ii) f is square integrable on the interval $[a_-, a_+]$.

In order to construct explicit solutions of the family F_2 (13) for linear and quadratic potentials (57)–(59) we have to solve equations (50). After some tedious calculations we obtain the following solutions:

(1) linear potential (57):

$$\psi(x,t) = f(x+at^2)\exp\left\{-\mathrm{i}\left(atx + \frac{2a}{3}t^3 + C\right)\right\} \tag{76}$$

(2) harmonic oscillator potential (58):

$$\psi(x,t) = (\cos 2\sqrt{2}at)^{-\frac{1}{4}} f\left(\frac{x^2}{\cos 2\sqrt{2}at}\right) \tag{77}$$

$$\times \exp\left\{-\mathrm{i}\left(\frac{a}{2\sqrt{2}}x^2\tan 2\sqrt{2}at + \frac{\mu_1}{4}\ln\cos 2\sqrt{2}at + C\right)\right\} \tag{78}$$

(3) anti-harmonic oscillator potential (59):

$$\psi(x,t) = (\cosh 2\sqrt{2}at)^{-\frac{1}{2}} f\left(\frac{x^2}{\cosh 2\sqrt{2}at}\right) \tag{79}$$

$$\times \exp\left\{\mathrm{i}\left(\frac{a}{2\sqrt{2}}x^2\tanh 2\sqrt{2}at - \frac{\mu_1}{4}\ln\cosh 2\sqrt{2}at + C\right)\right\}. \tag{80}$$

Here f is an arbitrary twice continuously differentiable function and C is an arbitrary constant.

5. Conclusion

As mentioned above, Lie symmetries of the considered DG equations are not extensive enough to provide their linearizability by means of local transformations. PDEs (9), (10) prove to be integrable because of infinite *non-local* symmetries admitted. Take, as an example, system (38). It has been decoupled into a system of two linear first-order PDEs (42) by means of non-local transformations of dependent and independent variables (16), (40), (41). It is well known (see, e.g., [22]) that any linear first-order PDE admits an infinite parameter Lie transformation group. Consequently, system (42) possesses an infinite local symmetry. But after being rewritten in the initial variables $(x, t, r(x,t), s(x,t))$ it becomes non-local and cannot be found by using the infinitesimal Lie algorithm (for more details about non-local symmetries of linear and nonlinear PDEs see [39, 40]).

In the case involved, the existence of non-local symmetry was indicated by a change of local symmetry of the DG equation when the parameters were specified to be (7), (8). These additional local symmetries form the top of the 'iceberg', the main part of which consists of non-local symmetries enabling us to integrate the corresponding DG equations.

Since we have the formulae for general solutions of systems of PDEs (9), (10) it is only natural to apply these to analyse the initial value problem for these systems, which is important for a physical interpretation of the equations. For example, using formula (36) it is not difficult to prove that the initial value problem

$$\mathrm{i}\psi_t = \left\{(\mu_1 - \mathrm{i})\,\mathrm{Im}\,\frac{\psi_{xx}}{\psi} + 2\left(\mathrm{Im}\,\frac{\psi_x}{\psi}\right)^2 - \mu_1\,\mathrm{Im}\left(\frac{\psi_x}{\psi}\right)^2\right\}\psi$$
$$\psi(x,0) = r_0(x)\exp\{\mathrm{i}s_0(x)\}$$

where $r_0, s_0 \in C^\infty(\mathbb{R},\mathbb{R})$ are arbitrary functions such that $\mu_1 s_0(x) \neq -\infty$, has a unique solution given by the formula (36) where $u(x,t)$ is a solution of the initial value problem for the heat equation

$$u_t + \mu_1 u_{xx} = 0 \qquad u(x,0) = \exp\left\{\frac{1}{\mu_1}s_0(x)\right\}$$

and the function $f(y)$ reads

$$f(y) = R(h(y))\exp\left\{-\frac{1}{\mu_1}S(h(y))\right\}.$$

Here $h(y)$ is determined implicitly by the relation

$$y = \int_0^{h(y)} \exp\left\{\frac{2}{\mu_1}s_0(\tau)\right\}\mathrm{d}\tau.$$

But for the DG equation (9) an analysis of the initial value problem is complicated due to the complex structure of its general solution.

The method of integration of DG equations developed in the present paper for the case of one space variable can be extended to the physically more interesting case of three spatial dimensions. The principal idea of such an extension is the utilization of the generalized Euler–Ampére transformations of the space $(t, \boldsymbol{x}, u(t,\boldsymbol{x}), u_t(t,\boldsymbol{x}), \mathrm{grad}\,u(t,\boldsymbol{x}))$ suggested in [41]. The above transformations were used to study compatibility and to construct a general solution of the four-dimensional nonlinear d'Alembert–eikonal system. This problem is under investigation now and will be a topic of our future publications.

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